Modeling and Identification of a Fractional-Order Discrete-Time Laguerre-Hammerstein System

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1 Introduction

Nonlinear block-oriented systems, including the Hammerstein, Wiener and feedback-nonlinear ones have attracted considerable research interest both from the industrial and academic environments [1, 2, 3, 4]. On the other hand, it is well known that orthonormal basis functions (OBF) have proved to be useful in identification and control of dynamical systems, including nonlinear block-oriented systems [5, 6, 7, 8]. In particular, an inverse OBF (IOBF) modeling approach has been effective in identification of a linear dynamic part of the Hammerstein system [5]. The approach provides the so-called separability in estimation of linear and nonlinear submodels [6], thus eliminating the bilinearity issue detrimentally affecting e.g. the ARX-based modeling schemes. The IOBF modeling approach is continued to be efficiently used here to model a linear fractional-order dynamic part of the Hammerstein system.

Recently, fractional-order dynamics have been given a huge research interest, mostly for linear systems [9, 10, 11, 12, 13, 14, 15, 16, 17].

Discrete-time fractional-order OBF-based modeling is a new research area and there is a few papers on the topic that has up to date been available [18, 19, 20, 21, 22]. Those papers illustrate that fractional-order discrete Laguerre filters can be very effective in modeling of dynamical systems.

A fractional-order Hammerstein system has been elegantly analyzed and identified in Ref. [23]. However, a computational burden of the approach is very high, in fact prohibitively high in adaptive estimation and control.

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This paper presents a new, simple strategy for Hammerstein system identification, which is a combination of the inverse-OBF modeling concept and fractional-order generalization of discrete Laguerre filters. The effective combination gives rise to the introduction of a powerful method for identification of the fractional-order Hammerstein system.

2 Fractional-Order Discrete-Time Difference

A simple generalization of the familiar Grünwald-Letnikov difference [11] is the fractional difference (FD) in discrete time t, described by equation [9, 13, 14, 24]

$$\Delta^{\alpha}x(t) = \sum_{j=0}^{t} P_j(\alpha)x(t)q^{-j} = x(t) + \sum_{j=1}^{t} P_j(\alpha)x(t)q^{-j} \quad t = 0, 1, \ldots \tag{1}$$

where $\alpha \in (0, 2)$ is the fractional order, q^{-1} is the backward shift operator and

$$P_j(\alpha) = (-1)^j \gamma_j(\alpha) \tag{2}$$

with

$$\gamma_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & j > 0 \end{cases} \tag{3}$$

Note that each element in Eqn. (1) from time t back to 0 is nonzero so that each incoming sample of the signal $x(t)$ increases the complication of the model equation. In the limit, with $t \to +\infty$, we end up with computational explosion. Therefore in [25], truncated or finite fractional difference (FFD) has been considered for practical,

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feasibility reasons. Finite fractional difference (FFD) is defined as

$$\Delta^{\alpha}x(t,J) = x(t) + \sum_{j=1}^{J} P_j(\alpha)x(t)q^{-j} \tag{4}$$

where $J = \min(t,\bar{J})$ and $\bar{J}$ is the upper bound for j when $t > \bar{J}$.

In this paper, we assume that α is known.

Remark 1. Possible accounting for the sampling period T when transferring from the Grünwald-Letnikov continuous-time derivative to the Grünwald-Letnikov discrete-time difference results in dividing the right-hand side of Eqns. (1) and (4) by T^{α}. Operating without T^{α} as in the sequel corresponds to putting $T = 1$ or to the substitution of $P_j(\alpha)$ for $\frac{P_j(\alpha)}{T^{\alpha}}$, $j = 0, \ldots, t$.

3 Fractional-Order Discrete-Time Laguerre Filters

A classical (or integer-order, or "regular") OBF model of a dynamical system, or shortly, OBF system, can be presented in form

$$y(t) = \sum_{i=1}^{K} C_i L_i(q)u(t) + e(t) \tag{5}$$

where $u(t)$ and $y(t)$ are the system input and output, respectively, $L_i(z)$ and C_i, $i = 1, \ldots, K$, are orthonormal transfer functions and weighting parameters, respectively and $e(t)$ is the output error. In case of use of discrete Laguerre filters we have

$$L_i(z) = \frac{k}{z-P}\left(\frac{-Pz+1}{z-P}\right)^{i-1} \quad i = 1,\ldots,K \tag{6}$$

where $k = \sqrt{1-P^2}$ and P is a dominant pole. In the sequel, we limit our interest to the practically justified case of $P > 0$. The unknown parameters C_i, $i = 1, \ldots, K$, can be easily estimated using e.g. Recursive Least Squares (RLS) or Least Mean Squares (LMS) algorithms formalized in a linear regression fashion [26]. In our examples, RLS estimation is used. Pursuing an optimal Laguerre pole P_{opt} has been well established [7, 8, 27, 28, 29].

The Laguerre filters presented in Eqn. (6), can be factorized to the form [25, 20, 21]

$$L_i(q^{-1}) = G_L(q^{-1})\left(G_R(q^{-1}) - P\right)^{i-1} \quad i = 1,\ldots,K \tag{7}$$

with

$$G_L(q^{-1}) = \frac{kq^{-1}}{1-Pq^{-1}} \tag{8}$$

$$G_R(q^{-1}) = \frac{k^2q^{-1}}{1-Pq^{-1}} = kG_L(q^{-1}) \tag{9}$$

and the consecutive filter outputs being $y_L(t) = G_L(q^{-1})u(t)$ and $y_R^i(t) = G_R(q^{-1})U_i(t)$, $i = 1,\ldots,K-1$, with

$$U_i(t) = \begin{cases} y_L(t) & i = 1 \\ y_R^{i-1}(t) - PU_{i-1}(t) & i = 2,\ldots,K \end{cases} \tag{10}$$

The two filters can also be described as

$$G_L^f: \quad \Delta y_L(t) = (P-1)y_L(t)q^{-1} + ku(t)q^{-1} \tag{11}$$

$$G_R^f: \quad \Delta y_R^i(t) = (P-1)y_R^i(t)q^{-1} + k^2U_i(t)q^{-1} \tag{12}$$

where $\Delta y_L(t) = y_L(t) - y_L(t-1)$ and similar is $\Delta\, y_R{}^i(t)$, $i = 1, \ldots, K$.

The outstanding value of the factorization (7) of the expression (6) is that $G_L(q^{-1})$ and $G_R(q^{-1})$ are the first-order filters that can be easily adopted to the fractional-order form. The fraction-formalized filters $G_L^f(q^{-1})$ and $G_R^f(q^{-1})$ can now be described as

$$G_L^f: \quad \Delta^{\alpha} y_L(t) = (P-1)y_L(t)q^{-1} + ku(t)q^{-1} \tag{13}$$

$$G_R^f: \quad \Delta^{\alpha} y_R^i(t) = (P-1)y_R^i(t)q^{-1} + k^2U_i(t)q^{-1} \tag{14}$$

where $U_i(t)$ is as in Eqn. (10). Finally, the outputs from the FD versions of the $G_L(q^{-1})$ and $G_R(q^{-1})$ filters can be obtained as

$$G_L^f: \quad y_L(t) = (P-1)y_L(t)q^{-1} + ku(t)q^{-1} - \sum\nolimits_{j=1}^{t} P_j(\alpha)y_L(t)q^{-j} \tag{15}$$

$$G_R^f: \quad y_R^i(t) = (P-1)y_R^i(t)q^{-1} + k^2U_i(t)q^{-1} - \sum\nolimits_{j=1}^{t} P_j(\alpha)y_R^i(t)q^{-j} \tag{16}$$

The outputs for FFD versions of the $G_L(q^{-1})$ and $G_R(q^{-1})$ filters can be calculated as

$$G_L^f: \quad y_L(t) = (P-1)y_L(t)q^{-1} + ku(t)q^{-1} - \sum\nolimits_{j=1}^{J} P_j(\alpha) y_L(t) q^{-j} \tag{17}$$

$$G_R^f: \quad y_R^i(t) = (P-1)y_R^i(t)q^{-1} + k^2 U_i(t)q^{-1} - \sum\nolimits_{j=1}^{J} P_j(\alpha) y_R^i(t) q^{-j} \tag{18}$$

Remark 2. Possible accounting for the sampling period T when transferring from the Grünwald-Letnikov continuous-time derivative to the Grünwald-Letnikov discrete-time difference results in multiplication of the two first components at the right-hand sides of Eqns. (17) and (18) by T^{α}.

Finally, the output (5) from the fractional-order Laguerre system is computed as

$$y(t) = \sum_{i=1}^{K} C_i U_i(t) \tag{19}$$

with $U_i(t)$ calculated in Eqn. (10).

4 System description

4.1 Non-fractional case [6, 8, 30]

The Hammerstein system (Fig. 1) consists of two cascaded elements, where the first one is a nonlinear memoryless gain and the second is a linear dynamic model. The whole Hammerstein system can be described by the equation

$$y(t) = G(q)[f(u(t)) + e_H(t)] = G(q)[v(t) + e_H(t)] \tag{20}$$

where $G(q)$ models a dynamic linear part, $f(.)$ describes a nonlinear function, $v(t)$ is the unmeasured output of the nonlinear part and $e_H(t)$ is the error/distur-bance term. An alternative output error/disturbance formulation is also possible. In case of use of the inverse OBF (IOBF) concept to model a linear dynamic part, the Hammerstein model equation can be presented in inverse form [6, 30]

$$\hat{G}^{-1}(q)\hat{y}(t) = v(t) \tag{21}$$

or

$$R(q)\hat{y}(t) = v(t) \tag{22}$$

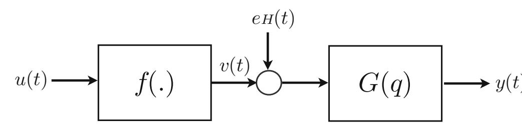

Fig. 1 Hammerstein system

where $R(q)$ is the inverse of the system model $\hat{G}(q)$. In the IOBF concept, the inverse $R(q)$ of the system is modeled using OBF. An OBF modeling approach can now be applied to equation (22) instead of (21) and finally we can present equation (20) in the following form [6, 30]

$$y(t) + \sum_{i=1}^{M} C_i L_i(q) y(t) = \frac{1}{r_0} v(t-d) + e_1(t) \tag{23}$$

where $e_1(t)$ is the equation error, d is the time delay of the system, and r_0 is the leading coefficient of $R(q)$.

The nonlinear part of the Hammerstein system $f(.)$ can be approximated e.g. with the polynomial expansion

$$v(t) = f(u(t)) = a_1 u(t) + a_2 u^2(t) + \ldots + a_m u^m(t) \tag{24}$$

with the coefficient a_1 put to 1 without loss of generality [30].

Combining equations (23) and (24) we arrive at the equation describing the model output $\hat{y}(t)$ of the whole Hammerstein system

$$\hat{y}(t) = -\sum_{i=1}^{M} C_i L_i(q) y(t) + \frac{1}{r_0} \sum_{i=1}^{m} a_i u^i(t-d) \tag{25}$$

with linear and nonlinear submodels separated from each other. Now that the bilinearity effect has been avoided thanks to the separation of the submodels, Eqn. (25) can be easily presented in the linear regression form.

4.2 Fractional-order case

We assume now that a linear dynamics is of fractional order. In order to embed the fractional-order Laguerre filters of Section 3 in the IOBF framework the following important remark is due.

Remark 3. It is essential that the Laguerre filters are, in the IOBF framework, driven by $y(t)$. This means that in order to calculate the fractional-order output equation (20) in the IOBF fashion we have to substitute $y(t)$ for $u(t)$ in Eqns. (13), (15) and (17).

Equation (25) can now be rewritten in form

$$\hat{y}(t) = -\sum_{i=1}^{M} C_i U_i(t) + \frac{1}{r_0}\sum_{i=1}^{m} a_i u^i(t-d) \qquad (26)$$

which can be presented in a linear regression form

$$\hat{y}(t) = \varphi^T(t)\Theta \qquad (27)$$

where $\Theta^T = [C_1 \quad \ldots \quad C_M \quad \beta_1 \quad \ldots \quad \beta_m]$ and $\varphi^T(t) = [-U_1(t) \quad \ldots \quad -U_M(t) \quad u(t-d) \quad \ldots \quad u^m(t-d)]$ with $\beta_i = a_i/r_0$ and $U_i(t)$, $i = 1, \ldots, M$ driven by $y(t)$ as in Remark 3. Now, the parameters Θ can be easily estimated using e.g. the RLS algorithm (or its adaptive version ALS).

5 Simulation Experiments

Example 1 Consider a discrete fractional-order Hammerstein system, with a static nonlinearity $f(u(t)) = u^3(t)$ and a fractional-order dynamic part described in state-space

$$\Delta^{\alpha} x(t+1) = A_f x(t) + Bu(t), \qquad (28)$$

$$y(t) = Cx(t) + Du(t) \qquad (29)$$

with

$$A_f = \begin{bmatrix} -0.4 & -0.03 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = [0 \quad 0.23],\ D = [0],$$

$$\alpha = 0.5$$

The dynamic part is modeled by an FFD-based fractional-order Laguerre model, with $P = 0.49$, $M = 8$, $m = 3$ and various implementation lengths of the FFD approximation ($\bar{J}$). MSPE is used to evaluate the accuracy of modeling. Selected results are presented in Table 1.

Fig. 2 presents the results of modeling in terms of (indistinguishable) time plots of the actual and modeled outputs of the Hammerstein system for some random input signal.

It can be concluded from Fig. 2 and Table 1 that the introduced fractional-order Laguerre-Hammerstein model can be very effective in modeling of the class of block-oriented nonlinear systems. However, to obtain high modeling accuracies we have to use high implementation lengths of the FFD approximation. This inconvenience can be essentially reduced by making use of our computationally more efficient approximations to FD, that is AFFD,

Table 1 MSPE for Hammerstein system with the FFD-based Laguerre model

$\bar{J}$	50	200	500	1000	5000
MSPE	1.637	0.578	0.182	0.107	8.79e-2

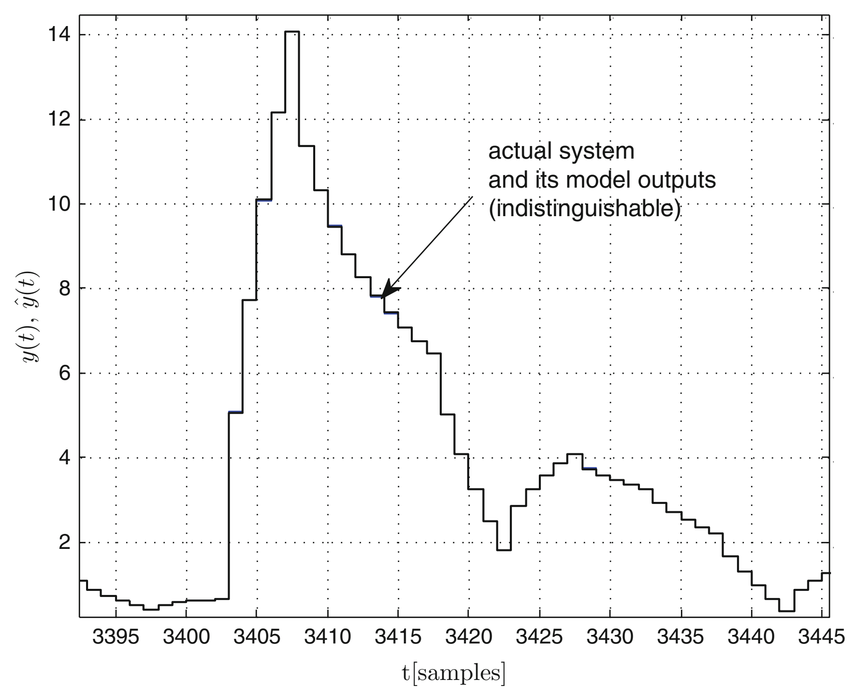

Fig. 2 Time plots of actual and modeled output of the Laguerre-Hammerstein system

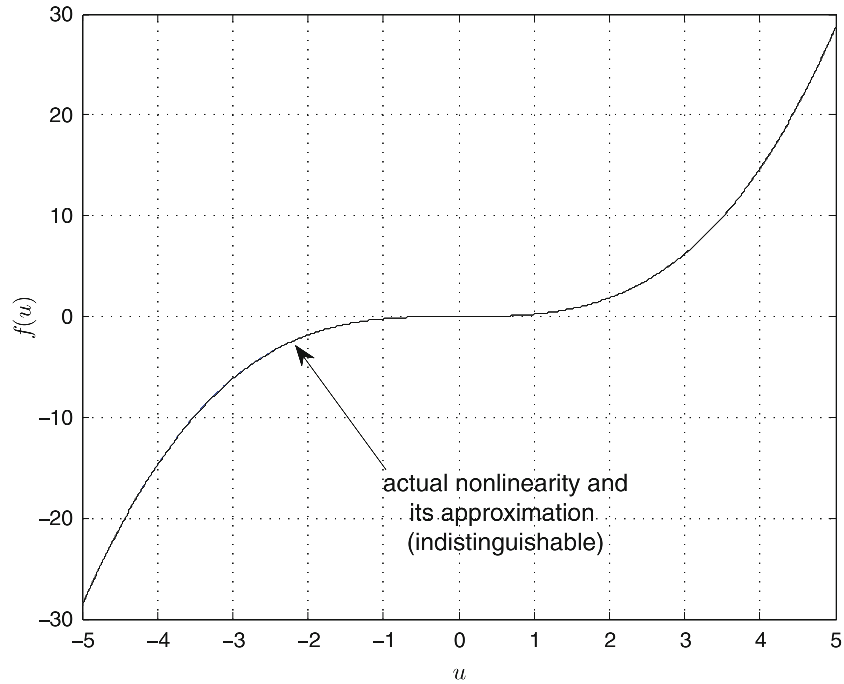

Fig. 3 Nonlinear static characteristic and its model

Table 2 MSPE for Hammerstain system with the FFD-based Laguerre model

$\bar{J}$	50	200	500	1000	5000
MSPE	534.0	533.4	532.9	532.7	532.7

PFFD [31], FLD and, in particular, FFLD [18]. Plots of the actual nonlinear static characteristic and its reconstruction presented in Fig. 3 confirm a very good identification performance. Right the same is with reconstruction of the linear part, in terms of indistinguishable respective impulse responses.

Example 2. Consider the fractional-order nonlinear system as in Example 1, with the zero mean disturbance $e_H(t)$ and $var(e_H(t)) = 0.1$.

Although the MSPEs are now visibly higher, the modeled nonlinear characteristic is, again, indistinguishable from the original one. However, a model of the linear dynamic subsystem is less precisely reconstructed and this is caused by the specific location of the noise $e_H(t)$. Still, the dynamic model accuracy is very good here.

6 Conclusion

The paper has presented a new simple, analytical solution to the nonlinear identification problem for the Hammerstein system using fractional-order Laguerre-based models. We have demonstrated that a combination of the inverse OBF modeling concept and fractional-order Laguerre filters can provide high-perfor-mance identification of fractional-order nonlinear systems. Simulation examples show that in both deterministic and stochastic cases, low prediction errors and accurate reconstructions of both nonlinear and linear parts of the system have been obtained for the introduced models.

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